A CANONICAL DEFINITION FOR THE PLANAR BERNARDI/ROTOR ROUTING ACTION

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Abstract. This is a simplified extract of the paper *The sandpile group of a trinity and a canonical definition for the planar Bernardi action* by the same set of authors. This paper contains no new material. It aims to be more accessible than the above mentioned paper by concentrating only on the notions needed to give the canonical definition, and omitting the greater picture.

1. Introduction

For an undirected graph, the sandpile group is a finite Abelian group whose order equals the number of spanning trees of the graph. Free transitive group actions of the sandpile group on the spanning trees have recently been an active topic of investigation. Two such classes of group actions are the rotor-routing actions [12] and the Bernardi actions [2]. In [7, 8, 2], some remarkable and somewhat mysterious properties of these group actions were uncovered.

Both group actions are defined using the same auxiliary data, namely a ribbon structure (or combinatorial embedding) and a fixed vertex (which is called the base point). It was first asked by Ellenberg on mathoverflow whether the rotor routing action of a sandpile group is independent of the root vertex. He also asked if one can think of the spanning trees as $\mathrm{Pic}^d(G)$ in some appropriate sense.

Inspired by his question, Chan, Church and Grochow [7] proved that the rotor-routing action is independent of the base point if and only if the ribbon structure is planar (that is, the graph is embedded into the plane). Chan, Glass, Macauley, Perkinson, Werner and Yang [8] showed that moreover, in the planar case, the rotor-routing action is compatible with planar duality (in a well-defined sense).

Baker and Wang [2] proved analogous results about the Bernardi action, i.e., that the Bernardi action is independent of the base point if and only if the ribbon structure is planar, and in the planar case, the Bernardi action is compatible with planar duality. They also showed that in the planar case, the Bernardi and rotor-routing actions coincide. Ding [10] and Shokrieh and Wright [18] showed that however for any nonplanar ribbon structure, there exists a base point such that the rotor routing action and the Bernardi action differ.

TK was supported by a Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research C (no. 17K05244).

SL was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03930998). SL is grateful to the CAMPUS Asia program from which he was supported to visit Tokyo Institute of Technology as an exchange student.

LT was supported by the NSF grant DMS-1455272 and the National Research, Development and Innovation Office of Hungary – NKFIH, grants no. 128673 and 132488.

In [14], we gave a canonical (that is, base point free) definition for the Bernardi/rotor-rouing action in the case of planar graphs. In this paper, we aim to give the simplest possible presentation of that canonical definition. This canonical definition is essentially the following: Any plane graph G has a medial graph D_E that is an Eulerian digraph, and its vertices correspond to the edges of G. We show that the sandpile group $\mathrm{Pic}^0(D_E)$ of the medial graph is canonically isomorphic to the sandpile group $\mathrm{Pic}^0(G)$. Moreover, we show that the spanning trees $\mathcal{T}(G)$ form a system of representatives of the equivalence classes in $\mathrm{Pic}^{|V|-1}(D_E)$. Clearly, $\mathrm{Pic}^0(D_E)$ acts on $\mathrm{Pic}^{|V|-1}(D_E) = \mathcal{T}(G)$ by addition. We define our canonical action as the composition of the action of $\mathrm{Pic}^0(D_E)$ with the canonical isomorphism between $\mathrm{Pic}^0(D_E)$ and $\mathrm{Pic}^0(G)$. Then we show that this action agrees with both the planar Bernardi action, and the planar rotor-routing action.

Hence in some sense, we can answer the question of Ellenberg in the affirmative: Indeed the set of spanning trees can be thought of as a coset of the sandpile group, the only twist is that it is natural to see them so for the "edge sandpile group" $\mathrm{Pic}^0(D_E)$.

We do not present the results of [14] in full generality. In fact, it turned out in [14] that the canonical definition can be naturally given for a slightly larger class of graphs than planar undirected graphs: for Eulerian directed graphs embedded in a balanced way. That latter result uncovered hidden symmetries of the Bernardi action, but we do not address these results here, as they need extra definitions and make the presentation lengthier.

2. Preliminaries

2.1. **Basic definitions.** Throughout this paper, we assume all graphs and directed graphs to be connected. We allow loops and multiple edges. For a directed graph (digraph) D, we denote the outdegree of a node v by $d^+(v)$. For two disjoint sets of nodes U and W, we denote by $d(U,W)$ the number of directed edges having their tail in U and their head in W. In particular, for vertices u and v, we let $d(u,v)$ denote the number of edges pointing from u to v.

A subgraph of an undirected graph is called a spanning tree if it is connected and cycle-free. For an undirected graph G, we denote the set of spanning trees by $\mathcal{T}(G)$. A subgraph of a digraph is called an in-arborescence with root r if we get a tree by forgetting its orientation, and r is reachable on a directed path from each vertex.

For an undirected graph, a ribbon structure is the choice of a cyclic ordering of the edges around each vertex. If a graph is embedded into an orientable surface, the embedding gives a ribbon structure using the positive orientation of the surface, and conversely, for any ribbon graph there exists a closed orientable surface of minimal genus so that the graph embeds into it, giving the particular ribbon structure. For us, the most important case is the case of the graphs embedded into the plane (plane graphs). For an edge xy of the graph, we denote by xy^+ the edge following xy at x according to the ribbon structure. For a digraph, a ribbon structure is a choice of a cyclic ordering of the union of in- and out-edges around each vertex.

The Laplacian matrix of a digraph is the following matrix $L_D \in \mathbb{Z}^{V\times V}$:

$$L_D(u,v) = \begin{cases} -d^+(v) & \text{if } u = v, \\ d(v,u) & \text{if } u \neq v. \end{cases}$$

Let us introduce notations for some special vectors in $\mathbb{Z}^{|V|}$. By $\mathbf{0}$, we denote the vector with all coordinates equal to zero, while by $\mathbf{1}$ the vector with all coordinates equal to one. For a set $S \subseteq V$, we let $\mathbf{1}_S$ denote the characteristic vector of S, i.e. $\mathbf{1}_S(v) = 1$ for $v \in S$ and $\mathbf{1}_S(v) = 0$ otherwise.

2.2. **The sandpile group.** In this subsection we give the definition of the sandpile group. As later on we will need the sandpile group of Eulerian digraphs, we give the definition for this broader case.

For an Eulerian digraph $D = (V, A)$, we denote by $\mathrm{Div}(D)$ the free Abelian group on V. For $x \in \mathrm{Div}(D)$ and $v \in V$, we use the notation $x(v)$ for the coefficient of v. We refer to x as a chip configuration, and to $x(v)$ as the number of chips on v. We use the notation $\deg(x) = \sum_{v \in V} x(v)$, and call $\deg(x)$ the *degree* of x. We also write $\mathrm{Div}^d(D) = \{x \in \mathrm{Div}(D) : \deg(x) = d\}$.

We call two chip configurations x and y *linearly equivalent* if there exists $z \in \mathbb{Z}^V$ such that $y = x + L_D z$. We use the notation $x \sim y$ for linear equivalence. Notice that, as for Eulerian digraphs we have $L_D \mathbf{1} = \mathbf{0}$, we can suppose that z has nonnegative elements and $z(v) = 0$ for some $v \in V$. Note also that linearly equivalent chip configurations have equal degree. We denote the linear equivalence class of a chip configuration x by $[x]$.

There is an interpretation of linear equivalence using the so-called chip-firing game. In this game, a step consists of firing a node v. The firing of v decreases the number of chips on v by the outdegree of v, and increases the number of chips on each neighbor w of v by $d(v, w)$. It is easy to check that the firing of v changes x to $x + L_D \mathbf{1}_v$. Hence x is linearly equivalent to y if and only if there is a sequence of firings that transforms x to y.

The Picard group of a digraph is the group of chip configurations factorized by linear equivalence: $\mathrm{Pic}(D) = \mathrm{Div}(D)/_\sim$. This is an infinite group. We will be interested in the subgroup corresponding to zero-sum elements, which is called the sandpile group.

Definition 2.1 (Sandpile group)**.** For an Eulerian digraph D, the sandpile group is defined as $\mathrm{Pic}^0(D) = \mathrm{Div}^0(D)/\sim$.

It is easy to see that $\mathrm{Pic}(D) = \mathrm{Pic}^0(D) \times \mathbb{Z}$. The sandpile group is a finite group.

We will use the notation $\mathrm{Pic}^d(D)$ for the set of equivalence classes of $\mathrm{Pic}(D)$ consisting of chip configurations of degree d.

If we have an undirected graph, we can apply the above definitions to the bidirected version of the graph, that is, where we substitute each undirected edge by two oppositely directed edges. It is a well-known fact with many different proofs, that for an undirected graph G, the order of $\mathrm{Pic}^0(G)$ is equal to the number of spanning trees of G.

2.3. **Rotor-routing.** In this subsection we recall the definition of the rotor-routing action, and the rotor-routing game following [12]. Rotor routing is more natural to be defined on a directed ribbon graph, hence we first give the definition for digraphs. At the end on the subsection, we explain how undirected graphs fit into the picture.

Let D be a ribbon digraph. For an out-edge e pointing away from a vertex v, we denote by e^+ the next out-edge of v according to the ribbon structure. Let v_0 be a fixed vertex of D that we call the *root*.

A *rotor configuration* is a function ϱ that assigns to each vertex $v \neq v_0$ an out-edge with tail v. We call $\varrho(v)$ the *rotor* at v. A configuration of the rotor-routing game is a pair (x, ϱ), where $x \in \mathrm{Div}(D)$ and ϱ is a rotor configuration. Given a configuration (x, ϱ), a *routing* at vertex v results in the configuration (x', ϱ'), where ϱ' is the rotor configuration with

$$\varrho'(u) = \begin{cases} \varrho(u) & \text{if } u \neq v, \\ \varrho(u)^+ & \text{if } u = v, \end{cases}$$

and $x' = x - \mathbf{1}_v + \mathbf{1}_{v'}$ where v' is the head of $\varrho'(v)$. (That is, we turn the rotor at v by one step, then place a chip from v to the head of the new rotor edge of v.) A step of the rotor-routing game is to take a vertex with a positive number of chips, and perform a routing at that vertex.

The rotor-routing action (with root v_0) is an action of $\mathrm{Pic}^0(D)$ on the in-arborescences of D rooted at v_0. For an in-arborescence A of D rooted at v_0 and an element $x \in \mathrm{Pic}^0(D)$ we denote by $r_{v_0}(x, A)$ the rotor-routing action (with root v_0) of x on the arborescence A, that we define in the next paragraph.

We first define the action of chip configurations of the form $\mathbf{1}_v - \mathbf{1}_{v_0}$. Let A be an in-arborescence rooted at v_0. We can think of A as a rotor-configuration, since each vertex $v \neq v_0$ has exactly one out-edge in A. Play a rotor-routing game started from $(\mathbf{1}_v - \mathbf{1}_{v_0}, A)$ until the chip reaches v_0. In other words, in this moment, the configuration of the game will be $(\mathbf{0}, \varrho)$ for some ϱ. It is proved in [12], that we eventually reach such a configuration, moreover, the rotor configuration ϱ will be another arborescence A' at this moment. (Notice that in this situation, the game is deterministic. It can happen that during the game, in some steps the rotor-configuration is not an arborescence, but when the chip eventually reaches v_0, it will be. See more in [12].) Then take $r_{v_0}(\mathbf{1}_v - \mathbf{1}_{v_0}, A) = A'$.

Note that (equivalence classes of) chip configurations of the form $\mathbf{1}_v - \mathbf{1}_{v_0}$ generate $\mathrm{Pic}^0(D)$. The action of a general $x \in \mathrm{Pic}^0(D)$ is defined linearly. Holroyd et. al. proves [12] that this, this is a well-defined group action of $\mathrm{Pic}^0(D)$ on the in-arborescences rooted at v_0, moreover, it is free and transitive.

For an undirected graph G, one defines the rotor-routing action as the action of the bidirected graph corresponding to G (where the ribbon structure of the bidirected graph also correspond to the ribbon structure of G).

In the undirected (bidirected) case, an in-arborescence can be identified with an undirected spanning tree, by forgetting the orientations. Hence we can say that the rotor-routing action of an undirected graph (with root v_0) acts on the spanning trees. More formally: Let G be an undirected graph, and let D be the bidirected graph corresponding to it. For a spanning tree T of G, let us denote by $\mathrm{arb}_{v_0}(T)$ the unique in-arborescence rooted at v_0 in D whose underlying undirected graph is T. Also, for an in-arborescence A of D, let $\mathrm{tree}(A)$ be the spanning tree of G obtained by forgetting the orientations of A. Then with these notations, the rotor-routing action on the spanning trees is: $r_{v_0}(x, T) = \mathrm{tree}(r_{v_0}(x, \mathrm{arb}_{v_0} T))$.

We will use this convention for undirected graphs, and say that (for an undirected graph) the rotor-routing action (with any root) acts on the set of spanning trees.

2.4. **The Bernardi action.** Let us turn to the Bernardi action for undirected graphs. For an (undirected) graph G, the Bernardi action is an action of $\mathrm{Pic}^0(G)$ on the spanning trees of G. To define it, we first need the definition of the Bernardi bijection.

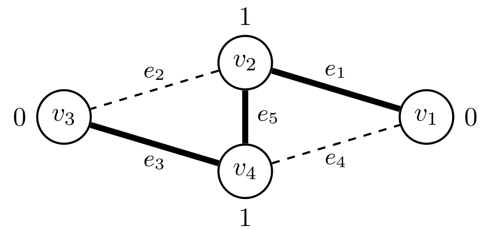

FIGURE 1. An example of the tour of a spanning tree and the Bernardi bijection. Let the ribbon structure be the one induced by the positive (counterclockwise) orientation of the plane, $b_0 = v_1$, and $b_1 = v_2$. The tour of the tree of thick edges is $v_1e_1, v_2e_2, v_2e_5, v_4e_3, v_3e_2,\ v_3e_3, v_4e_4, v_4e_5, v_2e_1, v_1e_4$. The Bernardi bijection gives the break divisor indicated by the numbers.

The Bernardi bijection depends on a ribbon structure of G and on a fixed vertex b_0 of G and a fixed edge b_0b_1 incident to b_0. Using this data, to any spanning tree T, one can now associate a traversal of the graph, which is called the *tour* of T. (This process was introduced by Bernardi [3, 4].) The tour of T is the following sequence of node-edge pairs: The current node at the first step is b_0, and the current edge is b_0b_1. If the current node is x, the current edge is xy, and $xy \notin T$, then the current node of the next step is x, and the current edge of the next step is xy^+. If the current node is x, the current edge is xy, and $xy \in T$, then the current node of the next step is y, and the current edge of the next step is yx^+. The tour stops when b_0 would once again become current node with b_0b_1 as current edge. (For an example, see Figure 1). Bernardi proved the following:

Lemma 2.2. [3, Lemma 5] *In the tour of a spanning tree T, each edge xy of G becomes current edge twice, in one case with x as current node, and in the other case with y as current node.*

For a graph $G = (V, E)$, a chip configuration $x \in \mathrm{Div}(G)$ is a break divisor if there exists a spanning tree T of G such that $E - T = \{e_1, \dots e_g\}$ and there is a bijection between the edges $\{e_1, \dots e_g\}$ and the chips of x such that each chip sits on one of the endpoints of the edge assigned to it. The following property makes break divisors very useful.

Theorem 2.3. [1] *For an undirected graph $G = (V, E)$, the set of break divisors form a system of representatives of linear equivalence classes of* $\mathrm{Pic}^{|E|-|V|+1}(G)$. *In other words, each linear equivalence class of degree $|E| - |V| + 1$ contains exactly one break divisor.*

The Bernardi bijection is a bijection between spanning trees and break divisors. It associates a chip configuration to any spanning tree by dropping a chip at the current vertex each time a nonedge of T first becomes current (see again Figure 1, also [2]). It is clear that this process produces a break divisor for any spanning tree. Bernardi [3] and Baker and Wang [2] prove that this is in fact a bijection between the spanning trees of G and the break divisors. We denote this bijection by β_{b_0,b_1}.

For a graph G, the sandpile group $\mathrm{Pic}^0(G)$ acts on $\mathrm{Pic}^{|E|-|V|+1}(G)$ by addition: For $x \in \mathrm{Pic}^0(G)$ and $z \in \mathrm{Pic}^{|E|-|V|+1}(G)$, we put $x \cdot z = x + z$. Since by Theorem

2.3, the break divisors give a system of representatives for $\mathrm{Pic}^{|E|-|V|+1}(G)$, we can think of this natural action as the action of $\mathrm{Pic}^0(G)$ on the break divisors: for $x \in \mathrm{Pic}^0(G)$ and a break divisor f, we have $x \cdot f = x \oplus f$, where by $x \oplus f$ we denote the unique break divisor in the linear equivalence class of $x + [f]$, which exists by Theorem 2.3. We call this group action the *sandpile action*.

The Bernardi action is defined by pulling the sandpile action of $\mathrm{Pic}^0(G)$ from the break divisors to the spanning trees by using a Bernardi bijection. We denote it by b_{b_0,b_1}. Hence $b_{b_0,b_1}(x,T) = \beta^{-1}_{b_0,b_1}(x \oplus \beta_{b_0,b_1}(T))$, where $x \in \mathrm{Pic}^0(G)$ and T is a spanning tree of G.

2.5. **Previous results on the rotor-routing and Bernardi actions.** It was asked by Ellenberg whether the rotor-routing action of an undirected graph is independent of the root vertex. Chan, Church and Grochow [7] proved that the rotor-routing action of an undirected ribbon graph is independent of the root if and only if the ribbon structure is planar. Moreover, Chan, Glass, Macauley, Perkinson, Werner and Yang also proved [8], that for planar graphs, the rotor-routing action is also compatible with planar duality. (We explain the meaning of this statement in the next subsection.)

Similar results were obtained for the Bernardi action by Baker and Wang [2]. They proved that the Bernardi action does not depend on the choice of b_1, moreover, it is independent of b_0 if and only if the ribbon structure is planar. For planar graphs, Baker and Wang also proved the compatibility of the Bernardi action with planar duality. Moreover, they proved that for planar graphs, the Bernardi and the rotor routing actions agree.

In the nonplanar case, Ding [10] and Shokrieh and Wright [18] showed that for any nonplanar ribbon structure, there exists a base point such that the rotor routing action and the Bernardi action differ.

2.6. **The meaning of compatibility with planar duality.** Let G be a planar graph, and G^* its planar dual. Then the edges of G are in one-to-one correspondence with the edges of G^*. For an edge e of G, let us denote by e^* the corresponding edge of G^*.

It is well-known that there is a canonical bijection between the spanning trees of a plane graph and its dual: To any spanning tree T of G, we can associate $T^* = \{e^* : e \notin T\}$, which is a spanning tree of G^*.

Also, for a planar graph G, there is a canonical isomorphism $i : \mathrm{Pic}^0(G) \to \mathrm{Pic}^0(G^*)$ (see [9]). Let us repeat the definition of i as given in [2]. Let G be a planar undirected graph. We need to fix an orientation $\overrightarrow{e}$ for each edge e. Now orient each edge e^* of G^* so that the corresponding edge $\overrightarrow{e}$ of G has to be turned in the negative direction to get the orientation of $\overrightarrow{e}^*$. For an edge $\overrightarrow{e}$ of G, let $\delta_{\overrightarrow{e}} \in \mathbb{Z}^V$ be the vector that has coordinate one on the head of $\overrightarrow{e}$, minus one on the tail of $\overrightarrow{e}$, and zero otherwise. For any $g \in \mathrm{Div}^0(G)$, one can find integers $\{a_{\overrightarrow{e}} : e \in E\}$ such that $\sum_{e\in E} a_{\overrightarrow{e}} \delta_{\overrightarrow{e}} = g$. Moreover, two collections of coefficients $\{a_{\overrightarrow{e}} : e \in E\}$ and $\{b_{\overrightarrow{e}} : e \in E\}$ give linearly equivalent chip configurations if and only if $\{a_{\overrightarrow{e}} - b_{\overrightarrow{e}} : e \in E\}$ can be written as the sum of an integer flow in G and an integer flow in G^*. Now for $[g] \in \mathrm{Pic}^0(G)$, the image $i([g])$ is defined as $[\sum_{e\in E} a_{\overrightarrow{e}} \delta_{\overrightarrow{e}^*}]$. It can be shown that this is a well defined mapping, which is an isomorphism, and it is independent of the orientation we chose. For more details, see [2] and its references.

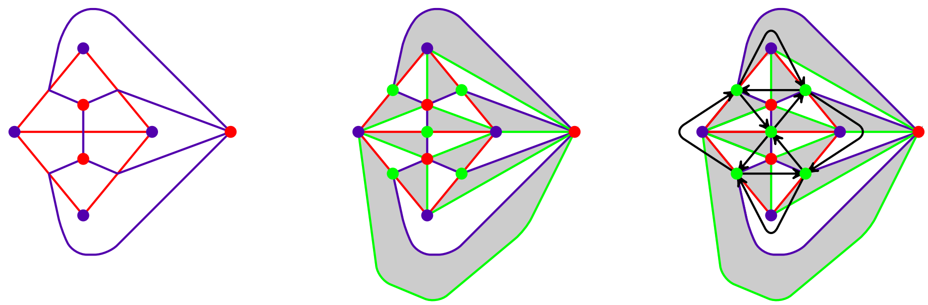

FIGURE 2. A plane graph and its dual (left panel), the corresponding trinity (middle panel), and the digraph D_E (right panel).

Then an action $a : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ is said to be compatible with planar duality, if for any $x \in \mathrm{Pic}^0(G)$ and spanning tree T, we have $a(x, T)^* = a(i(x), T^*)$.

3. Trinities and canonical isomorphisms

In this section, we recall the notion of trinities. Moreover, we define the "edge sandpile group", and using trinities, we introduce canonical isomorphisms between the sandpile groups of a planar graph, its dual, and their common edge sandpile group.

Definition 3.1 (Trinity)**.** A *trinity* is a triangulation of the sphere S^2 together with a three-coloring of the 0-simplices. (I.e., 0-simplices joined by a 1-simplex have different colors.) According to dimension, we will refer to the simplices as *points, edges*, and *triangles*. (See Figure 2 for an example.)

Trinities will be important for us because embedded planar graphs naturally yield them. (There are also trinities that do not come from planar graphs. These are considered in [14] but here we will concentrate on trinities that come from graphs.)

This is how we can construct a trinity from a planar ribbon graph. (See Figure 2 for an example.) Let G be a planar ribbon graph. Let V be the set of vertices of G, and color these vertices violet. Subdivide each edge of G by a new node. These new nodes are in one-to-one correspondence with the edges of G. Hence we denote the set of these nodes by E and color them emerald. Let us call the obtained bipartite graph G_R. Then place a red node in the interior of each region of G_R, and call the set of these nodes R (they correspond to the regions of the plane graph G). Traverse the boundary of each region of G_R and at each corner of the boundary, connect the emerald or violet node to the red node of the region. This way we get a three-colored triangulation of the surface. Let us color a triangle white if the violet, emerald and red vertices follow each other in positive cyclic order, and color it black otherwise.

Notice that the triangulation contains two further bipartite graphs (other than G_R). Let us call G_V the bipartite graph connecting vertices of R and E (this is the graph obtained by subdividing each edge of the planar dual G^* with a new node), and let us call G_E the bipartite graph connecting vertices of R and V.

We can also associate three directed graphs D_V, D_E and D_R to a trinity: The node set of D_V is V, and a directed edge points from $v_1 \in V$ to $v_2 \in V$ if a black triangle incident to v_1 and a white triangle incident to v_2 share their violet edge. D_E and D_R are defined analogously. Notice that if we obtained the trinity from a planar graph as described above, then D_V is the "bidirected" version of G (that is, each edge is substituted by two oppositely directed edges), and D_R is the bidirected version of G^*. D_E is generally not bidirected, but it is always Eulerian, as the indegree of any node $e \in E$ is the number of white triangles incident to it, and the outdegree is the number of black triangles incident to e, and these triangles alternate around e. We call D_E the medial (di)graph of G.

In this paper, we will always consider graphs G embedded into the plane. Hence we always imagine that the above described trinity is constructed for G. We will simultaneously think of edges of G in the ordinary sense (as curves connecting two vertices), and as an emerald node of the trinity. Moreover, we will think of edges of D_V as orientations of the edges of G.

It is well-known that for a planar graph G, $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(G^*)$ are canonically isomorphic [9]. That is, if we construct the above described trinity from the graph G, then $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_R)$ are canonically isomorphic. We next show that $\mathrm{Pic}^0(D_E)$ is also canonically isomorphic to $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_R)$.

3.1. **The isomorphism between** $\mathrm{Pic}^0(D_V)$**,** $\mathrm{Pic}^0(D_E)$ **and** $\mathrm{Pic}^0(D_R)$**.** We obtain natural isomorphisms between $\mathrm{Pic}^0(D_V)$, $\mathrm{Pic}^0(D_E)$, and $\mathrm{Pic}^0(D_R)$ using a group that we call the trinity sandpile group, and which appeared first in [6]. First we need some preparation.

Definition 3.2 ($\mathcal{A}$)**.** Let $\mathcal{A}$ be the free Abelian group on the set $V \cup E \cup R$. We describe the elements of $\mathcal{A}$ by vector triples (x_V, x_E, x_R), where $x_V \in \mathbb{Z}^V$, $x_E \in \mathbb{Z}^E$, and $x_R \in \mathbb{Z}^R$.

Definition 3.3 (white triangle equivalence)**.** Two elements of $\mathcal{A}$ are said to be white triangle equivalent if their difference can be written as an integer linear combination of characteristic vectors of white triangles. We denote white triangle equivalence by $\approx_W$.

Note that $\approx_W$ is indeed an equivalence relation. Now one can define a group by factorizing with white triangle equivalence. We call this group $\mathcal{A}_W$ the trinity sandpile group.

Definition 3.4 ($\mathcal{A}_W$, [6])**.** $\mathcal{A}_W = \mathcal{A}/_{\approx_W}$.

Blackburn and McCourt proved that $\mathcal{A}_W$ is isomorphic to the direct product of $\mathbb{Z}^2$ and $\mathrm{Pic}^0(D_V)$ [5]. But in fact it turns out there there is an even stronger relationship between the sandpile groups and the trinity sandpile group: the sandpile groups can be neatly embedded as subgroups of the trinity sandpile group.

Theorem 3.5. [14] *The equivalence classes of* $\mathcal{A}_W$ *containing at least one element of the form* $(x_V, \mathbf{0}, \mathbf{0})$ *with* $\deg(x_V) = 0$ *form a group isomorphic to* $\mathrm{Pic}^0(D_V)$*.*

The equivalence classes of $\mathcal{A}_W$ *containing at least one element of the form* $(\mathbf{0}, x_E, \mathbf{0})$ *with* $\deg(x_E) = 0$ *form a group isomorphic to* $\mathrm{Pic}^0(D_E)$*.*

The equivalence classes of $\mathcal{A}_W$ *containing at least one element of the form* $(\mathbf{0}, \mathbf{0}, x_R)$ *with* $\deg(x_R) = 0$ *form a group isomorphic to* $\mathrm{Pic}^0(D_R)$*.*

Proof. The statement of the theorem is implied by the following lemma, whose proof can be found in [14].

Lemma 3.6. *$x_V \sim y_V$ in D_V if and only if $(x_V, \mathbf{0}, \mathbf{0}) \approx_W (y_V, \mathbf{0}, \mathbf{0})$.*
$x_E \sim y_E$ in D_E if and only if $(\mathbf{0}, x_E, \mathbf{0}) \approx_W (\mathbf{0}, y_E, \mathbf{0})$.
$x_R \sim y_R$ in D_R if and only if $(\mathbf{0}, \mathbf{0}, x_R) \approx_W (\mathbf{0}, \mathbf{0}, y_R)$.

■

Remark 3.7. Theorem 3.5 is true for any trinity, not only for ones coming from planar graphs. (One can define the digraphs D_V, D_E and D_R for any trinity. In general, they will be Eulerian digraphs.) This is an example where concentrating on trinities obtained from graphs hides symmetries. The definition of (general) trinities is completely symmetric for the three color classes. However, if we consider trinities coming from planar graphs, then the color class E has special properties.

The above embedding of the three sandpile groups immediately gives isomorphisms between them.

Definition 3.8. Let $\varphi_{V\to E} : \mathrm{Pic}^0(D_V) \to \mathrm{Pic}^0(D_E)$ be defined by $\varphi_{V\to E}([x]) = [y]$ where $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$.

We will shortly prove that $\varphi_{V\to E}$ is well-defined (in particular, we can always find such a y).

Remark 3.9. Notice that $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$ is equivalent to $(x, y, \mathbf{0}) \approx_W (\mathbf{0}, \mathbf{0}, \mathbf{0})$. Hence $\varphi_{V\to E}([x]) = [y]$ can be witnessed by the linear combination of some white triangles, such that the linear combination of the characteristic vectors gives $(x, y, \mathbf{0})$.

We can analogously define $\varphi_{V\to R} : \mathrm{Pic}^0(D_V) \to \mathrm{Pic}^0(D_R)$ by $\varphi_{V\to R}([x]) = [z]$ where $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, \mathbf{0}, -z)$. We next show that these are well defined isomorphisms of the corresponding sandpile groups. These isomorhisms will play a fundamental role in the followings. Also, it will turn out, that $\varphi_{V\to R}$ generalizes the canonical isomorphsm i between the sandpile group of a graph and its dual (which was defined in Subsection 2.6). Note that the definition of $\varphi_{V\to R}$ is canonical, while the definition of i depended on an arbitrary orientation (even though the actual isomorphism was independent of it).

Theorem 3.10. *$\varphi_{V\to E}$ is well-defined and is an isomorphism between* $\mathrm{Pic}^0(D_V)$ *and* $\mathrm{Pic}^0(D_E)$.

Proof. We start with well-definedness. We claim that if $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$ and $(x', \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y', \mathbf{0})$ for $x \sim x'$ (in D_V), then $y \sim y'$ in D_E. Indeed, Lemma 3.6 implies $(x, \mathbf{0}, \mathbf{0}) \approx_W (x', \mathbf{0}, \mathbf{0})$, moreover, by the transitivity of $\approx_W$, we have $(\mathbf{0}, -y, \mathbf{0}) \approx_W (\mathbf{0}, -y', \mathbf{0})$, hence also $(\mathbf{0}, y, \mathbf{0}) \approx_W (\mathbf{0}, y', \mathbf{0})$. By Lemma 3.6 applied to D_E, this implies $y \sim y'$ (in D_E).

Now we show that for any $x \in \mathrm{Pic}^0(D_V)$ there exists $y \in \mathrm{Pic}^0(D_E)$ such that $(x, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, -y, \mathbf{0})$. If $x = \mathbf{0}$, then $y = \mathbf{0}$ is a good choice. If there exist a violet node v with $x(v) > 0$, then as the sum of chips in x is zero, there exists another violet node u with $x(u) < 0$. Choose a path in G_E connecting v with u (there exists a path between them because of the connectedness of G_E). Now give weights -1 and $+1$ alternately to the white triangles incident with the path such that the triangle incident to v gets coefficient -1 (see Figure 3). Then by parity, the triangle incident with u has coefficient $+1$. Adding the characteristic vectors of these triangles with these weights to $(x, \mathbf{0}, \mathbf{0})$, we decreased the sum of the absolute

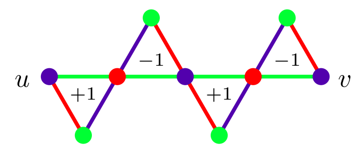

FIGURE 3. Illustration for the proof of Theorem 3.10.

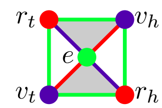

FIGURE 4. Illustration for the proof of Proposition 3.11.

values of the chips on violet vertices, while the number of chips on each red node remained 0. Continuing this way we reach a state with no chips on any violet or red node.

We have shown that $\varphi_{V\to E}$ is well defined. Interchanging the roles of V and E, the above two claims tell us that $\varphi_{V\to E}$ is injective and surjective.

It also follows immediately that $\varphi_{V\to E}$ is a homomorphism. Indeed, if we assume that $\varphi_{V\to E}([x_1]) = [y_1]$, $\varphi_{V\to E}([x_2]) = [y_2]$, and $\varphi_{V\to E}([x_1]+[x_2]) = [y_0]$, then

$$(\mathbf{0}, -y_0, \mathbf{0}) \approx_W (x_1+x_2, \mathbf{0}, \mathbf{0}) \approx_W (x_1, \mathbf{0}, \mathbf{0}) + (x_2, \mathbf{0}, \mathbf{0}) \approx_W \\ (\mathbf{0}, -y_1, \mathbf{0}) + (\mathbf{0}, -y_2, \mathbf{0}) \approx_W (\mathbf{0}, -(y_1+y_2), \mathbf{0}).$$

Since $\varphi_{V\to E}$ is well defined, it follows that $-y_0 \sim -(y_1+y_2)$, and hence $\varphi_{V\to E}([x_1+x_2]) = \varphi_{V\to E}([x_1]) + \varphi_{V\to E}([x_2])$. ■

By symmetry, $\varphi_{V\to R}$ is also well-defined, and is an isomorphism between $\mathrm{Pic}^0(D_V)$ and $\mathrm{Pic}^0(D_R)$.

Proposition 3.11. *For planar undirected graphs, $\varphi_{V\to R}$ agrees with i.*

Proof. Let G be a planar undirected graph, and take the corresponding trinity. We need to show that for an arbitrary orientation of G, and any $\{a_{\overrightarrow{e}} : e \in E\}$ we have $\varphi_{V\to R}([\sum_{e\in E} a_{\overrightarrow{e}} \delta_{\overrightarrow{e}}]) = [\sum_{e\in E} a_{\overrightarrow{e}} \delta_{\overrightarrow{e}^*}]$. For this, it is enough to show that for any $e \in E$, we have $\varphi_{V\to R}([\delta_{\overrightarrow{e}}]) = [\delta_{\overrightarrow{e}^*}]$. Let $v_h \in V$ be the head of $\overrightarrow{e}$ and let $v_t \in V$ be the tail of $\overrightarrow{e}$. Similarly, let $r_h \in R$ be the head of $\overrightarrow{e}^*$ and let $r_t \in R$ be the tail of $\overrightarrow{e}^*$. Let us also denote the emerald node corresponding to e by e. (See Figure 4.) We need to show that $(\mathbf{1}_{v_h} - \mathbf{1}_{v_t}, \mathbf{0}, \mathbf{0}) \approx_W (\mathbf{0}, \mathbf{0}, \mathbf{1}_{r_t} - \mathbf{1}_{r_h})$ or in other words, $(\mathbf{1}_{v_h} - \mathbf{1}_{v_t}, \mathbf{0}, \mathbf{1}_{r_h} - \mathbf{1}_{r_t}) \approx_W (\mathbf{0}, \mathbf{0}, \mathbf{0})$. But the relationship of $\overrightarrow{e}$ and $\overrightarrow{e}^*$ implies that v_h, r_h, e are the vertices of a white triangle, and v_t, r_t, e are vertices of another white triangle. Hence taking the first triangle with coefficient 1 and the second triangle with coefficient -1 proves the statement. ■

3.2. **Spanning trees are representatives of $\mathrm{Pic}^{|V|-1}(D_E)$.** To any spanning tree T of G, one can associate the characteristic vector of T, that is a vector in $\mathbb{Z}^E$ with coordinate 1 on $e \in T$ and coordinate 0 on $e \notin T$. By a slight abuse of notation, we denote this characteristic vector also by T.

Hence with this convention, $T \in \mathrm{Pic}^{|V|-1}(D_E)$ for any spanning tree. We claim that in fact, more is true.

Theorem 3.12. *Let G be a plane graph and D_E the medial graph of the trinity of G. Then the set of spanning trees of G gives a system of representatives of linear equivalence classes of $\mathrm{Pic}^{|V|-1}(D_E)$. In other words, for any chip configuration x_E on E with $\deg(x_E) = |V| - 1$, there is exactly one spanning tree $T \in \mathcal{T}(G)$ such that $T \sim x_E$ (where linear equivalence is meant for the graph D_E).*

Proof. For a set of edges $S \subset E$, let us denote by $V(S)$ the set of vertices incident to some edge of S.

First we show that any linear equivalence class of $\mathrm{Pic}(D_E)$ of degree $|V| - 1$ contains at most one spanning tree. Suppose for a contradiction that there exist two spanning trees $T, T' \in \mathcal{T}(G)$ such that $T \sim T'$ in D_E. This means that there exist $z \in \mathbb{Z}^E$ such that $T' = T + L_{D_E} z$. Since $L_{D_E}\mathbf{1} = \mathbf{0}$, we can suppose that z only has nonnegative elements, and it has a zero coordinate. Let $S = \{e \in E : z(e) = 0\}$. We will reach a contradiction by showing that $|T' \cap S| \geq |V(S)|$, which obviously cannot happen for a tree.

By the definition of S, we have $|T' \cap S| \geq |T \cap S| + d_{D_E}(E - S, S)$. Indeed, we can get from T to T' by firing (in D_E) each emerald node $e \in E$ exactly $z(e)$ times, in which case nodes of S do not fire, while each node of $E - S$ fires at least once. Thus S does not lose any chips, and it receives at least one chip through each edge of D_E leading from $E - S$ to S.

On the other hand, $|T \cap S| = |T| - |T \cap (E - S)| \geq |V| - |V(E - S)|$ since $|T| = |V| - 1$ and $|T \cap (E - S)| \leq |V(E - S)| - 1$. Hence

$$|T' \cap S| \geq |T \cap S| + d_{D_E}(E - S, S) \geq |V| - |V(E - S)| + d_{D_E}(E - S, S).$$

$|V| - |V(E - S)|$ is the number of vertices where all the incident edges are from S. We claim that $d_{D_E}(E - S, S)$ is at least the number of vertices that have incident edges from both S and $E - S$. Then, that implies $|T' \cap S| \geq |V(S)|$ which contradicts the fact that T' is a hypertree.

Hence let us show that $d_{D_E}(E - S, S)$ is at least the number of vertices that have incident edges from both S and $E - S$. The number $d_{D_E}(E - S, S)$ is the number of the edges of G_E (i.e., emerald edges) such that the black triangle incident to them has an emerald node from $E - S$ and the white triangle incident to them has an emerald node from S. Notice that for violet node v that has neighbors both from S and from $E - S$, there is at least one emerald edge incident to v with the above property. Indeed, if we look at the emerald neighbors of v in a positive cyclic order, there must be a time where after a neighbor from $E - S$, we see a neighbor from S. The emerald edge incident to v separating the triangles of these two neighbors will be appropriate. Hence $d_{D_E}(E - S, S)$ is at least the number of violet nodes that have neighbors from both S and $E - S$ in G_E.

With this we have proved that any linear equivalence class of $\mathrm{Pic}(D_V)$ of degree $|V| - 1$ contains at most one spanning tree.

As $\mathrm{Pic}^0(D_E)$ is isomorphic to $\mathrm{Pic}^0(G)$, and $|\mathrm{Pic}^{|V|-1}(D_E)| = |\mathrm{Pic}^0(D_E)|$, moreover, as by the matrix-tree theorem the number of spanning trees of G equals $|\mathrm{Pic}^0(G)|$, we conclude that we need to have exactly one spanning tree in each linear equivalence class of $\mathrm{Pic}^{|V|-1}(D_E)$. ■

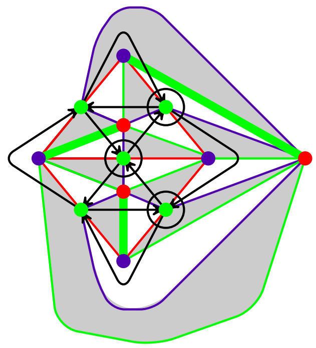

FIGURE 5. An illustration for the proof of Theorem 3.12. If S is the set of circled emerald nodes, then the edges of D_E leading from $E - S$ to S correspond to the thick emerald edges.

Remark 3.13. This theorem, too, has a generalization to arbitrary trinities. There, one needs to take an appropriate generalization of spanning trees. These turn out to be the so-called hypertrees from [13], and they are in fact a common generalization of spanning trees and break divisors. (More exactly, for a break divisor f, $\deg -\mathbf{1} - f$ is a hypertree. This is the reason that [14] sometimes has negated formulas with respect to this paper, as [14] uses hypertrees instead of break divisors.)

4. The canonical action and its compatibility with planar duality

Now we define an action of the sandpile group of a planar ribbon graph on the set of spanning trees. The definition uses only the embedding and needs no "base point" as auxiliary data. We also show (with an extremely short proof) that this action is compatible with planar duality.

Later on, we will show that this action agrees with the planar rotor-routing action and with the planar Bernardi action.

By Theorem 3.12, spanning trees correspond to $\mathrm{Pic}^{|V|-1}(D_E)$, hence we can define an action of $\mathrm{Pic}^0(D_E)$ on the spanning trees of G as the action of $\mathrm{Pic}^0(D_E)$ on its coset. In other words, for any spanning tree T, and $x \in \mathrm{Pic}^0(D_E)$, $x + T \in \mathrm{Pic}^{|V|-1}(D_E)$, hence there is exactly one spanning tree T' such that $x + T \sim T'$ (in D_E). Let us define $x \oplus T = T'$, and this is the action of x on T.

Now we can use the canonical isomorphism between $\mathrm{Pic}^0(G)$ and $\mathrm{Pic}^0(D_E)$ to pull this action to $\mathrm{Pic}^0(G)$.

Definition 4.1. Let G be a planar ribbon graph. For an element $x \in \mathrm{Pic}^0(G)$, and $T \in \mathcal{T}(G)$, we define $c(x, T) = \varphi_{V\to E}(-x) \oplus T$.

As the definition of the sandpile action and the isomorphism $\varphi_{V\to E}$ depended only on the embedding, $c : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ is indeed well defined.

Example 4.2. Figure 6 shows the computation of $c(x, T)$ for a concrete example. The first panel shows a spanning tree T and a chip configuration x. The second panel shows $(-x, \varphi_{V\to E}(-x), 0)$, as well as the linear combination of white triangles witnessing this. For clarity of the picture, we omitted the 0 coordinates. Of course

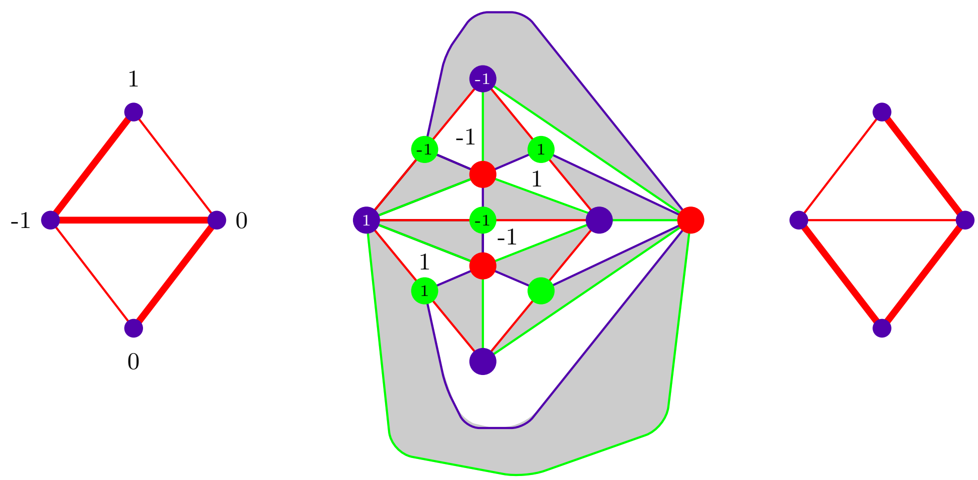

FIGURE 6. An example for the computation of the canonical action. See Example 4.2 for more details.

there are many representatives of $\varphi_{V\to E}(-x)$. We chose to draw the representative whose addition to T results in a spanning tree. The third panel shows this unique spanning tree T' in the linear equivalence class (in D_E) of $\varphi_{V\to E}(-x)+T$. Hence $c(x,T)=T'$.

For the sandpile action of $Pic^0(D_E)$ on $\mathcal{T}(G)$, it is extremely simple to prove compatibility with planar duality.

Theorem 4.3. *The sandpile action of $Pic^0(D_E)$ on the spanning trees of G and the sandpile action of $Pic^0(D_E)$ on the spanning trees of G^* are compatible with planar duality. In other words, for any $[x]\in \mathrm{Pic}^0(D_E)$ and $T\in\mathcal{T}(G)$, we have $(x\oplus T)^* = -x\oplus T^*$.*

Proof. Let $x\in Pic^0(D_E)$ and $T\in\mathcal{T}(G)$ be arbitrary and put $x\oplus T=T'$. Then $x\sim T'-T$. It is enough to show that $-x\sim (T')^*-T^*$, but this is easy to see: $(T')^*-T^* = T-T'\sim -x$. ■

The compatibility of the Bernardi action with planar duality is an immediate corollary.

Corollary 4.4. *The canonical action c is compatible with planar duality. In other words, for any $[x]\in Pic^0(G)$ and $T\in\mathcal{T}(G)$, we have $c([x],T)^* = c(\varphi_{V\to R}([x]),T^*)$.*

Proof. By definition,

$$c([x],T)=\varphi_{V\to E}([-x])\oplus T,$$

and

$$c(\varphi_{V\to R}([x]),T^*)=\varphi_{R\to E}(-\varphi_{V\to R}([x]))\oplus T^*.$$

It is easy to check from the definition that $\varphi_{R\to E}(-\varphi_{V\to R}(x))=\varphi_{V\to E}(x)$. Hence we see that $c(\varphi_{V\to R}([x]),T^*)=\varphi_{V\to E}([x])\oplus T^*$. Now by Theorem 4.3, $c([x],T)^* = (\varphi_{V\to E}([-x])\oplus T)^* = -\varphi_{V\to E}([-x])\oplus T^* = \varphi_{V\to E}([x])\oplus T^*$, where in the last step we used the obvious fact that $-\varphi_{V\to E}([-x])=\varphi_{V\to E}([x])$. Hence the two quantities indeed agree. ■

5. The planar rotor-routing action agrees with the canonical action

Theorem 5.1. *Let G be a plane graph. For any root v_0, chip-configuration x and spanning tree $T \in \mathcal{T}(G)$, $r_{v_0}(x,T) = \varphi_{V\to E}([-x]) \oplus T$, i.e. the rotor-routing action on G with root v_0 coincides with the canonical action. Consequently, the rotor-routing action of G is independent of the root, and is compatible with planar duality.*

It will be more convenient for us to work in the directed model. That is, we substitute the plane graph G with the bidirected graph D_V, and the spanning tree T with the arborescence $\mathrm{arb}_{v_0}(T)$ (that we think of as a subgraph of D_V).

We will need the following technical result from [19], that gives a way check if $r_{v_0}(x,A) = A'$ for in-arborescences A and A'.

Proposition 5.2. [19, Proposition 3.16] *If x is any chip configuration, A and A' are in-arborescences rooted at v_0, and we can get $(\mathbf{0}, A')$ from (x, A) by playing a rotor-routing game, then $x_{v_0}A = A'$.* ■

We need the following simple claim on in-arborescences, whose proof can be found for example in [14].

Claim 5.3. *For any two in-arborescences A and A', there exist a sequence of arborescences $A = A_0, A_1, \dots, A_k = A'$ such that A_{i+1} is obtained from A_i by adding and removing one edge.*

Proof of Theorem 5.1. Take the trinity obtained from G, and let us call $G = D_V$ in the followings. Fix our root v_0 (which is a violet node of the trinity). Let us fix a red node b_0 and an emerald node b_1 such that $v_0b_0b_1$ is a black triangle.

We need to show that for any $x \in \mathrm{Pic}^0(G) = \mathrm{Pic}^0(D_V)$ and spanning tree $T \in \mathcal{T}(G)$, we have $\mathrm{tree}(r_{v_0}(x, \mathrm{arb}_{V_0}(T))) = \varphi_{V\to E}([-x]) \oplus T$. By the simply transitive property of the rotor-routing action, it is enough to show instead that if for spanning trees $T, T' \in \mathcal{T}(G)$ we have $\mathrm{tree}(r_{v_0}(x, \mathrm{arb}_{v_0}(T))) = T'$ for some $x \in \mathrm{Pic}^0(D_V)$, then for this x, we also have $\varphi_{V\to E}([-x]) \oplus T = T'$.

Let $A = \mathrm{arb}_{v_0}(T)$ and $A' = \mathrm{arb}_{v_0}(T')$. By Claim 5.3 it is enough to consider the case when A' can be obtained from A by removing an arc and adding one. As A and A' are both in-arborescences, this means that an arc vv' is removed, and an arc vv'' is added. Suppose that in the ribbon structure of D_V the out-edges at v follow each other in the order $vv' = vu_0, vu_1, \dots, vu_k = vv''$. Then by Proposition 5.2, $[x] = [k\mathbf{1}_v - \mathbf{1}_{u_1} - \cdots - \mathbf{1}_{u_k}]$, since by performing k routings at v from the configuration (x, A), we arrive at $(\mathbf{0}, A')$, moreover, A and A' are both arborescences.

Now let us find $\varphi_{V\to E}([x])$. This is the equivalence class of a y such that $(x, y, 0) \approx_W (0,0,0)$. See Figure 7 for an illustration of the following arguments. Notice that G_V is the planar dual graph of D_V (with the exception that D_V is oriented, and G_V is not). Let r_ie_i be the edge of G_V dual to (the oriented edge) vu_i for $i = 0, \dots k$. Then $r_0, e_0, r_1, e_1, \dots, r_k, e_k$ follow each other in this order on the boundary of the face of G_V corresponding to v. By taking the white triangle vr_ie_{i-1} with coefficient one and the white triangles of the form $e_ir_iu_i$ with coefficient -1, for each $i = 1, \dots, k$, we obtain $(x, \mathbf{1}_{e_0} - \mathbf{1}_{e_k}, 0) \approx_W (0,0,0)$. Hence $\varphi_{V\to E}([x]) = [\mathbf{1}_{e_0} - \mathbf{1}_{e_k}]$.

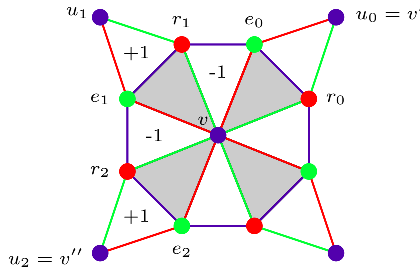

FIGURE 7. An illustration for the proof of Theorem 5.1.

Notice that e_0 is the edge that we get by forgetting the orientation of vv', and e_k is the edge that we get by forgetting the orientation of vv''. Hence $T' = T + e_0 - e_k$. Thus, indeed, $\varphi_{V\to E}([x]) \oplus T = T'$, finishing the proof. ■

6. The planar Bernardi action agrees with the canonical action

As Baker and Wang [2] showed, the Bernardi action of a planar graph agrees with the rotor-routing action of the same graph. Hence it follows that the cannonical action also agrees with the Bernardi action. However, the identity of the canonical action and the Bernardi action b_{b_0,b_1} can also be proved directly, without referring to the results of Baker and Wang. This would be a slightly (though not much more) longer proof as for the rotor routing action, hence we only cite here the main lemma without proof.

Remark 6.1. Note that in [14], the following theorem has $\varphi_{V\to E}(x)$ instead of $\varphi_{V\to E}(-x)$. This is so, because [14] considers the Bernardi bijection β_{b_0,b_1} as a bijection between spanning trees and hypertrees, and this paper (as well as Baker and Wang) considers the Bernardi bijection as a bijection between spanning trees and break divisors. Hence the Bernardi bijection of [14] would be $\deg -1 - \beta_{b_0,b_1}$ in this papers notations. This difference explains the difference in signs. The reason that [14] considers hypertrees instead of break divisors is that the Bernardi bijection can be generalized to arbitrary trinities if one considers hypertrees.

Theorem 6.2. [14] *For a balanced plane digraph with ribbon structure coming from the positive orientation of the plane, and any choice of $\{b_0, b_1\}$, the Bernardi bijection commutes with the sandpile actions. That is, the following diagram is commutative for any $x \in \mathrm{Pic}^0(D_V)$.*

$$\begin{array}{ccc} B_V(G_R) & \xrightarrow[x]{} & B_V(G_R) \\ \uparrow{\scriptstyle \beta_{b_0,b_1}} & & \uparrow{\scriptstyle \beta_{b_0,b_1}} \\ B_E(G_R) & \xrightarrow{\varphi_{V\to E}(-x)} & B_E(G_R) \end{array}$$

Recall that the definition of the Bernardi action is $b_{b_0,b_1}(x,T) = \beta^{-1}_{b_0,b_1}(x \oplus \beta_{b_0,b_1}(T))$. From the Theorem, it immediately follows that $b_{b_0,b_1}(x,T) = \varphi_{V\to E}(-x) \oplus T$.

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